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Lorentz-invariant Hamiltonian and Riemann hypothesis

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Abstract. We have given some arguments that a two-dimensional Lorentz-invariant Hamiltonian may be relevant to the Riemann hypothesis concerning zero points of the Riemann zeta function. Some eigenfunction of the Hamiltonian corresponding to infinite-dimensional representation of the Lorentz group have many interesting properties. Especially, a relationship exists between the zero zeta-function condition and the absence of trivial representations in the wavefunction. We also give a heuristic argument for the validity of the hypothesis.

The Riemann hypothesis (RH) [1–3] is one of the long-standing problems in the number theory. The Riemann's zeta function $\zeta(z)$ for a complex variable z is defined for $\operatorname{Re} z > 1$ by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

and for other values of z by its analytic continuation. It is well known that $\zeta(z)$ is zero for negative even integer values of z, i.e. $z = -2, -4, -6, \ldots$, while all other non-trivial zeros of $\zeta(z)$ must lie in the strip $0 < \operatorname{Re} z < 1$. It has been conjectured that all non-trivial zeros of $\zeta(z)$ actually lie on the critial line $\operatorname{Re} z = \frac{1}{2}$. This RH is important in number theory, since its validity can answer some questions concerning distributions of the prime numbers.

It has been suggested by many authors that the problem may be related to eigenvalue spectra of a self-adjoint operator H in some Hilbert space, although any such H has not been found so far. This view has been strengthened by the works of Odlyzko [4] and of others (see e.g. [5, 6] and references therein) that the statistical distribution of zero points of $\zeta(z)$ is consistent to a high degree with the law of the Gaussian unitary ensemble of random-matrix theory [5], which is expected for spectra of complex Hamiltonians. Moreover, this fact is also found to be related to the phenomenon of quantum chaos [6, 7]. A widely held opinion among many authors is that the validity of the RH with its associated Hamiltonian, if it exists, will shed light on quantum chaos and vice versa.

The purpose of this paper is to show the existence of a one-parameter family of complex Hamiltonians which seems to be intimately connected with the problem. Moreover, these Hamiltonians are invariant under two-dimensional Lorentz transformation, a fact which will be of some intrinsic interest in its own right.

We start from the following integral representation [8] of $\zeta(z)$:

$$\zeta(z) = \frac{1}{(1-2^{1-z})\Gamma(z)} \int_0^\infty \mathrm{d}t\, \frac{t^{z-1}}{1+\exp t} \qquad (\operatorname{Re} z > 0) \tag{1}$$

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so that any non-trivial zero of $\zeta(z)$ must satisfy

$$\int_0^\infty \mathrm{d}t\, \frac{t^{z-1}}{1+\exp t} = 0 \qquad (1 > \operatorname{Re} z > 0). \tag{2}$$

We call equation (2) the zero zeta-function condition (ZZFC). Also in view of the identity

$$2^{1-z}\Gamma(z)\zeta(z)\cos\left(\frac{\pi}{2}z\right) = \pi^z\zeta(1-z) \tag{3}$$

we may restrict ourselves to consideration of only the half sector $1 > \operatorname{Re} z \geqslant \frac{1}{2}$ instead of $1 > \operatorname{Re} z > 0$ for the ZZFC. If we can show that the assumption of $1 > \operatorname{Re} z > \frac{1}{2}$ for $\zeta(z) = 0$ will lead to a contradiction, this will prove the RH.

Suppose now that a Hamiltonian H is Hermitian in a Hilbert space, so that we have

$$\langle H\phi|\psi\rangle = \langle\phi|H\psi\rangle \tag{4}$$

for wavefunctions ϕ and ψ. For a complex z satisfying ZZFC, i.e. equation (2), we set

$$z = \tfrac{1}{2} + \mathrm{i}\lambda. \tag{5}$$

If H possessses an eigenfunction ϕ_0 with the eigenvalue λ, i.e. if we have

$$H\phi_0 = \lambda\phi_0 \tag{6}$$

then equation (4) with $\psi = \phi = \phi_0$ will give $\lambda = \bar{\lambda}$ being real and hence $\operatorname{Re} z = \frac{1}{2}$, proving RH. The natural question is whether such a H exists or not. Although we did not really succeed in this aim, we have found some pairs (H, ϕ_0) satisfying the required conditions. The problem is that the ϕ_0 found is not normalizeable, indicating that the spectrum of H is perhaps continuous rather than discrete. If so, the proposed Hamiltonian will not be directly relevant to the RH. However, there exists an intriguing connection between the ZZFC and the representation space of the Lorentz group under which H is invariant, and we can give a heuristic argument in favour of the validity of RH. These facts suggest that the present formulation may not be directly but indirectly relevant to the problem.

Let $\phi(x, y)$ and $\psi(x, y)$ be functions of two real variables x and y. We introduce the inner product by

$$\langle\phi|\psi\rangle = \int_{-\infty}^{\infty}\mathrm{d}x\int_0^\infty \mathrm{d}y\, \bar{\phi}(x, y)\psi(x, y). \tag{7}$$

Hereafter, $\bar{\phi}(x, y)$, for example, stands for the complex conjugate of $\phi(x, y)$. Note that the ranges of the integrations are $\infty > x > -\infty$ for x but $\infty > y \geqslant 0$ for y. Consider a family of second-order differential operators given by

$$H = \frac{\partial^2}{\partial x\partial y} + \mathrm{i}\beta y\frac{\partial}{\partial y} + \mathrm{i}(1-\beta)x\frac{\partial}{\partial x} + \frac{\mathrm{i}}{2} \tag{8}$$

for real parameter β. We note first that H is complex rather than real and second that it contains a purely imaginary constant term $\mathrm{i}/2$ whose presence is crucial for the hermiticity property of H, as we will see below. By a simple calculation, it is easy to find

$$\overline{(H\phi)}\psi - \bar{\phi}(H\psi) = \frac{\partial}{\partial x}J_1 + \frac{\partial}{\partial y}J_2 \tag{9a}$$

where we have set

$$J_1 = \frac{1}{2}\left(\frac{\partial\bar{\phi}}{\partial y}\psi - \bar{\phi}\frac{\partial\psi}{\partial y}\right) - \mathrm{i}(1-\beta)x\bar{\phi}\psi \tag{9b}$$

$$J_2 = \frac{1}{2}\left(\frac{\partial\bar{\phi}}{\partial x}\psi - \bar{\phi}\frac{\partial\psi}{\partial x}\right) - \mathrm{i}\beta y\bar{\phi}\psi. \tag{9c}$$

Note that the presence of the constant term $\mathrm{i}/2$ in the right-hand side of equation (8) is pivotal in enabling us to obtain equations (9). Integrating both sides of equation (9), we would find the hermiticity condition equation (4), if we could discard all partially integrated terms involving J_1 and J_2. From the explicit expressions of J_1 and J_2 given above, this would be possible, if ϕ and ψ or their derivatives with respect to x vanish at $y = 0$, and if ϕ and ψ as well as their derivatives decrease sufficiently rapidly for $x \to \pm\infty$ and $y \to \infty$. Of course, we have to study more carefully the question of the domain and range of H in order to establish the self-adjointness of H. However, the naive criteria given above suffices for this discussion. Especially, if ϕ satisfies equation (6), i.e.

$$H\phi = \lambda\phi \tag{10a}$$

with the boundary condition

$$\phi(x, 0) = 0 \tag{10b}$$

at $y = 0$ and if $\phi(x, y)$ decreases rapidly at infinity, this may lead to the RH in principle. We note that equation (10a) with $z = \frac{1}{2} + \mathrm{i}\lambda$ implies the validity of

$$\left\{\frac{\partial^2}{\partial x \partial y} + \mathrm{i}\beta y \frac{\partial}{\partial y} + \mathrm{i}(1-\beta)x\frac{\partial}{\partial x}\right\}\phi = -\mathrm{i}z\phi. \tag{11}$$

We have yet to meaningfully utilize the ZZFC in our formalism. Before going into detail, we will first, however, note the following property of the Hamiltonian. H as well as the inner product $\langle\phi|\psi\rangle$ are clearly invariant under the transformation

$$x \to \frac{1}{k}x \qquad \text{and} \qquad y \to ky \tag{12}$$

for any positive constant k. This invariance really reflects that of a two-dimensional Lorentz transformation. To understand it better, consider new variables u and v given by

$$x = u - v \qquad y = u + v. \tag{13}$$

The Hamiltonian H is then invariant under the SO(1, 1) Lorentz transformation

$$\begin{aligned} u \to u' &= (\cosh\theta)u + (\sinh\theta)v \\ v \to v' &= (\sinh\theta)u + (\cosh\theta)v \end{aligned} \tag{14}$$

for real constant θ, corresponding to the boost parameter $k = \exp\theta$. Because of the invariance, if ϕ satisfies $H\phi = \lambda\phi$, then so does $\phi(\frac{x}{k}, ky)$, and hence

$$\tilde{\phi}(x, y) = \int_0^\infty \frac{\mathrm{d}k}{k} f(k)\phi\left(\frac{x}{k}, ky\right) \tag{15}$$

for an arbitrary function, $f(k)$ also satisfies $H\tilde{\phi} = \lambda\tilde{\phi}$. In particular, any eigenfunction $\phi(x, y)$ of H may be regarded as a infinite-dimensional realization of the Lorentz group SO(1, 1).

After these preparations, we will now discuss solutions of the differential equation (11). We have found the following two families of solutions. Let $g(\xi)$ be an arbitrary function of a variable ξ which vanishes fast for $\xi \to \infty$. Then, we show first that

$$\phi(x, y) = \int_0^\infty \mathrm{d}t\, t^{z-1} \exp\{\mathrm{i}xt^{1-\beta}\} g(t + yt^\beta) \qquad (\mathrm{Re}\, z > 0) \tag{16}$$

where $\xi = t + yt^\beta$ is a solution of $H\phi = \lambda\phi$ with $z = \frac{1}{2} + \mathrm{i}\lambda$. In this connection if we change $x \leftrightarrow y$ and $\beta \leftrightarrow 1 - \beta$, it will furnish another solution. This can be proved as follows. For simplicity, set

$$G_0(x, y; t) = \exp\{\mathrm{i}xt^{1-\beta}\} g(t + yt^\beta) \tag{17a}$$

and note that G_0 satisfies a differential equation

$$\left\{\frac{\partial^2}{\partial x \partial y} + \mathrm{i}\beta y \frac{\partial}{\partial y} + \mathrm{i}(1-\beta)x\frac{\partial}{\partial x}\right\} G_0 = \mathrm{i}t\frac{\partial}{\partial t}G_0 \tag{17b}$$

as we can easily verify. Multiplying t^{z-1} and integrating over t from $t = \infty$ to $t = 0$, reproduces equation (11) if $\mathrm{Re}\, z > 0$. The special choice

$$g(\xi) = \frac{1}{1+\exp\xi}$$

is of particular interest. Then, the function f_0 given by

$$f_0(x, y) = \int_0^\infty \mathrm{d}t\, \frac{t^{z-1}}{1+\exp[t+yt^\beta]} \exp(\mathrm{i}xt^{1-\beta}) \tag{18a}$$

obeys

$$Hf_0 = \lambda f_0 \tag{18b}$$

although the ZZFC implies only

$$f_0(0, 0) = 0 \tag{19}$$

at the single point $x = y = 0$, instead of the desired boundary condition equation (10b) for arbitrary x. As we will see later, $f_0(x, y)$ is intimately related to the zeta function.

We can also find another class of solutions as follows. Let us now consider

$$G_1(x, y; u) = \frac{u^{\theta-1}}{(1-u)^\theta} \mathrm{e}^{-\mathrm{i}uxy} g(yu^\beta(1-u)^{1-\beta}) \tag{20}$$

for a constant θ with $\xi = yu^\beta(1-u)^{1-\beta}$ for arbitrary function $g(\xi)$. We can verify that G_1 satisfies the differential equation

$$\left\{\frac{\partial^2}{\partial x \partial y} + \mathrm{i}\beta y \frac{\partial}{\partial y} + \mathrm{i}(1-\beta)x\frac{\partial}{\partial x}\right\} G_1 - \mathrm{i}\frac{\partial}{\partial u}\{u(1-u)G_1\} = -\mathrm{i}\theta G_1. \tag{21}$$

Integrating equation (21) from $u = 1$ to $u = 0$, and assuming $1 > \mathrm{Re}\,\theta > 0$, we obtain

$$Hf_1 = \lambda_1 f_1 \tag{22a}$$

with

$$\theta = \tfrac{1}{2} + \mathrm{i}\lambda_1 \tag{22b}$$

if we set

$$f_1(x, y) = \int_0^1 \mathrm{d}u\, G_1(x, y, u). \tag{22c}$$

In order to obtain a solution which satisfies equation (10b), we let $x \to \frac{1}{k(t)}x$ and $y \to k(t)y$ for an arbitrary function $k(t)$ of a new variable t, and integrate equation (22c) on t after multiplying by $t^{z-1}(1+\exp t)^{-1}$. In this way, we generate a new family of solutions. In summary, the function

$$\phi_1(x, y) = \int_0^\infty \mathrm{d}t\, \frac{t^{z-1}}{1+\exp t} \int_0^1 \mathrm{d}u\, \frac{u^{\theta-1}}{(1-u)^\theta} \mathrm{e}^{-\mathrm{i}uxy} g(\xi) \tag{23a}$$

with

$$\xi = k(t)yu^\beta(1-u)^{1-\beta} \tag{23b}$$

for arbitrary functions $k(t)$ of t and $g(\xi)$ of ξ is a solution of

$$H\phi_1 = \lambda_1\phi_1 \qquad (\theta = \tfrac{1}{2} + \mathrm{i}\lambda_1). \tag{24}$$

Moreover, if $g(0)$ exists and is finite, then the ZZFC will give the desired boundary condition

$$\phi_1(x, 0) = 0 \tag{25}$$

for $1 \geqslant \beta \geqslant 0$, since $y = 0$ implies $\xi = 0$. Therefore, with the choice of $\theta = z$ and hence $\lambda_1 = \lambda$, the essential conditions equations (10) will be obeyed for $\phi = \phi_1$. However, a difficulty is that it appears to lead to $\langle\phi_1|\phi_1\rangle = \infty$ in general. This suggests that the spectrum of H is continuous rather than discrete. If so, H may have nothing directly to do with the RH. However, there exists an indication that this formulation may not be completely irrelevant to the problem as will be explained below.

The function $f_0(x, y)$ introduced by equation (18*a*) may also be related to RH for the following reason. We will first state, without proof, that there exist some constants C_0, C_1, C_2, and C_3 such that

$$|f_0(x, y)| \leqslant C_0 \tag{26a}$$

$$|f_0(x, y)| \leqslant C_1 y^{-\frac{1}{\beta}\,\mathrm{Re}\, z} \tag{26b}$$

$$|f_0(x, y)| \leqslant C_2 |x|^{-\frac{1}{1-\beta}\,\mathrm{Re}\, z} \tag{26c}$$

$$|f_0(x, y)| \leqslant C_3 |xy|^{-\mathrm{Re}\, z} \tag{26d}$$

under the assumption of

$$1 > \beta > 0. \tag{27}$$

If we have $\mathrm{Re}\, z > \frac{1}{2}$, then $\langle f_0|f_0\rangle$ is finite and the function $f_0(x, y)$ will furnish a infinite-dimensional *unitary* realization of the Lorentz group SO(1, 1) with or without the ZZFC. Moreover, if the ZZFC is assumed, we will have first, the orthogonality relation

$$\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \overline{G}(xy) f_0(x, y) = \int_{-\infty}^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \overline{G}(xy) f_0(x, y) = 0 \tag{28}$$

for an arbitrary function $G(\xi)$ with $\xi = xy$, which will vanish sufficiently fast for $\xi \to \infty$. Second, also under the ZZFC, the following is satisfied:

$$\int_0^\infty \frac{\mathrm{d}k}{k} f_0\left(\frac{x}{k}, ky\right) = 0. \tag{29}$$

We can show this as follows. We rewrite the left-hand side integral of equation (29) as

$$J = \int_0^\infty \frac{\mathrm{d}k}{k} f_0\left(\frac{x}{k}, ky\right) = \int_0^\infty \mathrm{d}t\, t^{z-1} \int_0^\infty \frac{\mathrm{d}k}{k} \frac{\exp(\mathrm{i}\frac{x}{k} t^{1-\beta})}{1 + \exp[t + kyt^\beta]}$$

and change the variable k into $k \to k' = kt^{\beta-1}$ to find

$$J = \int_0^\infty \mathrm{d}t\, t^{z-1} \int_0^\infty \frac{\mathrm{d}k'}{k'} \frac{\exp(\mathrm{i}x/k')}{1 + \exp[(1 + k'y)t]}.$$

Interchanging the order of the integrals and letting $t \to t' = (1 + k'y)t$, leads to

$$J = \int_0^\infty \frac{\mathrm{d}k'}{k'} \frac{\exp(\mathrm{i}x/k')}{(1 + k'y)^z} \int_0^\infty \mathrm{d}t' \frac{(t')^{z-1}}{1 + \exp t'}$$

which vanishes identically by the ZZFC.

Equation (28) can then be shown by changing the variable y into k and then letting $x \to \xi = kx$ to calculate

$$\int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}x\, \overline{G}(kx)\, f_0(x,k) = \int_0^\infty \mathrm{d}k \int_0^\infty \frac{\mathrm{d}\xi}{k}\, \overline{G}(\xi)\, f_0\left(\frac{\xi}{k}, k\right)$$
$$= \int_0^\infty \mathrm{d}\xi\, \overline{G}(\xi) \int_0^\infty \frac{\mathrm{d}k}{k}\, f_0\left(\frac{\xi}{k}, k\right)$$

which is zero by equation (29).

The condition (29) can be interpreted as implying that the infinite-dimensional representation space of SO(1, 1), spanned by $f_0(x, y)$ does *not* contain any singlet representation of the group. This is because the left-hand side of equation (29) is precisely the Lorentz-invariant component contained in the representation space, since it is invariant under $x \to \frac{1}{\alpha}x$ and $y \to \alpha y$ for any positive constant α. Then, the orthogonality relation equation (28) can be readily recognized as the one between two mutually inequivalent representations of SO(1, 1) since $G(xy)$ is clearly a Lorentz scalar. Such a relationship between the condition $\zeta(z) = 0$ and the absence of a trivial representation of SO(1,1) in $f_0(x, y)$ is quite intriguing and may indicate that $f_0(x, y)$ somehow plays a role in the RH.

We can now give a heuristic argument for the validity of RH as follows. Assuming $\mathrm{Re}\, z > \frac{1}{2}$ and $1 > \beta \geqslant 0$, we first calculate

$$J(k_1, k_2) = \int_{-\infty}^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \overline{f_0}\left(\frac{x}{k_1}, k_1 y\right) f_0\left(\frac{x}{k_2}, k_2 y\right) \tag{30}$$

for $k_1 > 0$ and $k_2 > 0$. Because of the Lorentz invariance, $J(k_1, k_2)$ is actually only a function of the variable $\xi = k_2/k_1$, i.e.

$$J(k_1, k_2) = J(1, k_2/k_1). \tag{31}$$

Rewriting

$$f_0\left(\frac{x}{k}, ky\right) = \frac{k^{\frac{z}{1-\beta}}}{1-\beta} \int_0^\infty \frac{\mathrm{d}s}{s}\, s^{\frac{z}{1-\beta}} \frac{\exp(\mathrm{i}xs)}{1 + \exp[(ks)^{\frac{1}{1-\beta}}(1 + \frac{y}{s})]}$$

we integrate first on the x-variable by using the Plancherel formula of Fourier integral theory. We then let $y \to y' = y/s$ and change the s-variable into $s' = s(1 + y')^{1-\beta}$ as in the derivation of equation (29). In this way, we calculate

$$J(k_1, k_2) = \frac{2\pi}{2\,\mathrm{Re}\, z - 1} \frac{(k_1)^{\frac{\bar{z}}{1-\beta}} (k_2)^{\frac{z}{1-\beta}}}{(1-\beta)^2} \int_0^\infty \frac{\mathrm{d}s}{s} \frac{s^{\frac{2}{1-\beta}\mathrm{Re}\, z}}{[1 + \exp(k_1 s)^{\frac{1}{1-\beta}}][1 + \exp(k_2 s)^{\frac{1}{1-\beta}}]}. \tag{32}$$

We can verify the validity of equation (31) by further letting $s \to s' = k_1 s$. The appearance of the denominator factor $(2\,\mathrm{Re}\, z - 1)^{-1}$ is due to the integration of

$$\int_0^\infty \mathrm{d}y' \frac{1}{(1 + y')^{z+\bar{z}}} = \frac{1}{2\,\mathrm{Re}\, z - 1}$$

which requires $\mathrm{Re}\, z > \frac{1}{2}$. Another way of deriving equation (32) is to integrate both sides of equation (9*a*) identifying $\phi(x, y) = f_0(\frac{x}{k_1}, k_1 y)$ and $\psi(x, y) = f_0(\frac{x}{k_2}, k_2 y)$. Since $H\phi = \lambda\phi$, and $H\psi = \lambda\psi$, the left-hand side of equation (9*a*) will then give

$$(\bar{\lambda} - \lambda) J(k_1, k_2) = \mathrm{i}(2\,\mathrm{Re}\, z - 1) J(k_1, k_2)$$

which also accounts for the factor $(2\,\mathrm{Re}\, z - 1)^{-1}$ in equation (32). We note that the integration of the right-hand side of equation (9*a*) will now give a non-zero contribution since $f_0(\frac{x}{k}, ky)$ and its derivative will not vanish at $y = 0$.

Let $\phi(k_1, k_2)$ be a function of k_1 and k_2 to be specified below and set

$$I = \int_0^\infty \mathrm{d}k_1 \int_0^\infty \mathrm{d}k_2\, \phi(k_1, k_2) J(k_1, k_2)$$
$$= \int_0^\infty \mathrm{d}k_1 \int_0^\infty \mathrm{d}k_2\, \phi(k_1, k_2) \int_{-\infty}^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \overline{f_0}\left(\frac{x}{k_1}, k_1 y\right) f_0\left(\frac{x}{k_2}, k_2 y\right). \quad (33)$$

Noting equation (31), we can calculate

$$I = \int_0^\infty \frac{\mathrm{d}\xi}{\xi} \psi(\xi) J(1, \xi) \quad (34)$$

where $\psi(\xi)$ is given by

$$\psi(\xi) = \xi \int_0^\infty \mathrm{d}k_1\, k_1\ \phi(k_1, \xi\ k_1). \quad (35)$$

Assuming $\phi(k_1, k_2) = \phi(k_2, k_1)$, we easily see that $\psi(\xi)$ must satisfy a functional equation

$$\psi\left(\frac{1}{\xi}\right) = \psi(\xi). \quad (36)$$

At any rate, equation (32) will lead to

$$I = \frac{2\pi}{2\,\mathrm{Re}\, z - 1} \int_0^\infty \mathrm{d}s\, \frac{s^{\bar{z}-1}}{1 + \exp s} \int_0^\infty \mathrm{d}t\, \frac{t^{z-1}}{1 + \exp t} \psi(s^{\beta-1} t^{1-\beta}) \quad (37)$$

after some calculations. Although we have to interchange orders of several integrals for the derivations of equations (32) and (37), these can be justified. We also note that I is real for real $\psi(\xi)$ because of equation (36).

We now choose $\phi(k_1, k_2)$ to be given by

$$\phi(k_1, k_2) = \frac{1}{k_1 k_2} (k_1 + k_2)^\delta \exp[-\alpha(k_1 + k_2)] \quad (38)$$

for positive numbers δ and α. We can then calculate

$$\psi(\xi) = \int_0^\infty \frac{\mathrm{d}k_1}{k_1} [(1 + \xi)k_1]^\delta \exp[-\alpha(1 + \xi)k_1]$$
$$= \int_0^\infty \mathrm{d}t\, t^{\delta-1} \exp(-\alpha t) = \alpha^{-\delta}\Gamma(\delta) \quad (39)$$

by changing k_1 into $t = (1 + \xi)k_1$. Since $\psi(\xi)$ is a constant, independent of ξ, then equation (37) implies $I = 0$ identically because of the ZZFC and hence we must have

$$\int_0^\infty \frac{\mathrm{d}k_1}{k_1} \int_0^\infty \frac{\mathrm{d}k_2}{k_2} (k_1 + k_2)^\delta \exp[-\alpha(k_1 + k_2)]$$
$$\times \int_{-\infty}^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \overline{f_0}\left(\frac{x}{k_1}, k_1 y\right) f_0\left(\frac{x}{k_2}, k_2 y\right) = 0 \quad (40)$$

for any $\delta > 0$ and $\alpha > 0$. We will now come to a murkier part. We first let $\delta \to 0$, and assume that we can interchange the order of the limit and integrals. If we can further interchange the order of k_1 and k_2 integrals and of x and y integrals, this will lead to

$$\int_{-\infty}^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, |A(x, y)|^2 = 0 \quad (41a)$$

with

$$A(x, y) = \int_0^\infty \frac{\mathrm{d}k}{k} \exp(-\alpha k) f_0\left(\frac{x}{k}, ky\right). \quad (41b)$$

Since the integrand of equation (41a) is non-negative, this requires

$$\int_0^\infty \frac{\mathrm{d}k}{k}\exp(-\alpha k) f_0\left(\frac{x}{k}, ky\right) = 0$$

for arbitrary positive α. This is clearly not possible. The contradiction can be avoided if the zero point z satisfies $\operatorname{Re} z = \frac{1}{2}$, i.e. the validity of the RH.

However, there is something wrong with the above reasoning. We have assumed only $\operatorname{Re} z > \frac{1}{2}$ so that the conclusion should also be valid for $\operatorname{Re} z = 1$. However, this cannot be correct for the following reason. Note that

$$f_0(0, 0) = \int_0^\infty \mathrm{d}t\, \frac{t^{z-1}}{1 + \exp t}$$

should be zero for $\operatorname{Re} z = 1$ at $z = 1 + (2\pi n\mathrm{i}/\log 2)$ for any non-zero integer value of n, since equation (1) implies otherwise that $\zeta(z)$ would possess poles at these points because of the denominator factor proportional to $1 - 2^{1-z}$. The most likely culprit for this dilemma is the indiscrimiate interchanges of orders of the limit $\delta \to 0$ and of integrals in the last stage of the demonstration. Unless we can give a rigorous justification for these interchanges only for $\frac{1}{2} < \operatorname{Re} z < 1$, the derivation given here must be regarded at best as a possible plausible argument which may or may not be correct. We hope that a more satisfactory method with a more directly appropriate Hamiltonian will be found in the future.

In conclusion, we make the following remark: in order to emphasize the dependence of $f_0(x, y)$ upon parameters β and z, we now explicitly write it as

$$F_0(x, y, z; \beta) = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}t\, \frac{t^{z-1}}{1 + \exp(t + yt^\beta)} \exp(\mathrm{i}xt^{1-\beta}) \tag{42}$$

so that $f_0 = \Gamma(z) F_0$ and it satisfies the differential equation

$$\left\{ \frac{\partial^2}{\partial x \partial y} + \mathrm{i}\beta y \frac{\partial}{\partial y} + \mathrm{i}(1-\beta)x\frac{\partial}{\partial x} \right\} F_0 = -\mathrm{i}z F_0 \tag{43a}$$

as well as

$$\frac{\partial}{\partial x} F_0(x, y, z; \beta) = \mathrm{i}F_0(x, y, z + 1 - \beta; \beta)\Gamma(z + 1 - \beta)/\Gamma(z). \tag{43b}$$

For the special cases $\beta = 0$ and $\beta = 1$, it reproduces the zeta function and its generalizations. For $\beta = 1$, we change the integration variable t into $t' = (1 + y)t$ and recall equation (1) to obtain

$$F_0(x, y, z; 1) = (1 - 2^{1-z})\zeta(z) \frac{\mathrm{e}^{\mathrm{i}x}}{(1 + y)^z}. \tag{44}$$

For $\beta = 0$, we calculate

$$F_0(x, y, z; 0) = \mathrm{e}^{-y}\Phi(-\mathrm{e}^{-y}, z, 1 - \mathrm{i}x) \tag{45}$$

where $\Phi(\xi, z, \eta)$ is the generalized zeta function defined by

$$\Phi(\xi, z, \eta) = \sum_{n=0}^\infty (\eta + n)^{-z}\xi^n \tag{46}$$

which converges for $|\xi| < 1$, $\eta \neq 0, -1, -2, -3, \ldots$. When we use the integral representation [8] of

$$\Phi(\xi, z, \eta) = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}t\, \frac{t^{z-1}}{1 - \xi \mathrm{e}^{-t}} \mathrm{e}^{-\eta t} \tag{47}$$

for Re $\eta > 0$, Re $z > 0$, $|\xi| \leqslant 1$, $\xi \neq 1$, and compare it with equation (42), we find equation (45). Since F_0 satisfies equation (43*a*), we see that Φ must be a solution of the differential equation

$$\left\{\xi \frac{\partial^2}{\partial\xi\,\partial\eta} + \eta\frac{\partial}{\partial\eta}\right\}\Phi(\xi, z, \eta) = -z\Phi(\xi, z, \eta) \tag{48}$$

which can also be easily verified from equation (46) but appears to have been overlooked in the literature.

In conclusion, we have attempted in this paper to present some arguments for the possible relevance of our Hamiltonian to the RH. Although they may not be the ultimate answer to the problem, there are at least some indications that they may be indirectly useful.

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